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Some Remarks on Incomplete Gamma Type Function $\gamma_*(\alpha, x_-)$

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Abstract. The incomplete gamma type function $\gamma_*(\alpha, x_-)$ is defined as locally summable function on the real line for $\alpha > 0$ by

$$\gamma_*(\alpha, x_-) = \begin{cases} \int_0^x |u|^{\alpha-1} e^{-u}\, du, & x \le 0, \\ 0, & x > 0 \end{cases}$$
$$= \int_0^{-x_-} |u|^{\alpha-1} e^{-u}\, du$$

the integral diverginig $\alpha \le 0$ and by using the recurrence relation

$$\gamma_*(\alpha+1, x_-) = -\alpha\gamma_*(\alpha, x_-) - x_-^\alpha e^{-x}$$

the definition of $\gamma_*(\alpha, x_-)$ can be extended to the negative non-integer values of α.

Recently the authors [8] defined $\gamma_*(-m, x_-)$ for $m = 0, 1, 2, \ldots$. In this paper we define the derivatives of the incomplete gamma type function $\gamma_*(\alpha, x_-)$ as a distribution for all $\alpha < 0$.

1. Introduction

The incomplete gamma function $\gamma(\alpha, x)$ is defined for $\alpha > 0$ and $x \ge 0$ by

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u}\, du \tag{1}$$

see [7], the integral diverging for $\alpha \le 0$. The incomplete gamma function can be defined for $\alpha < 0$ and $\alpha \ne -1, -2, -3, \ldots$ by using the recurrence formula

$$\gamma(\alpha+1, x) = \alpha\gamma(\alpha, x) - x^\alpha e^{-x}.$$

By regularization we have

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1}\Big[e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!}\Big]\, du + \sum_{i=0}^{m-1} \frac{(-1)^i x^{\alpha+i}}{(\alpha+i)i!}. \tag{2}$$

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for $-m < \alpha < -m+1$ and $x > 0$. It follows from the definition of gamma function that

$$\lim_{x\to\infty} \gamma(\alpha, x) = \Gamma(\alpha)$$

for $\alpha \neq 0, -1, -2, \ldots$, see [4, 6, 9].

In the following we let $\mathcal{N}$ be the neutrix [1, 4, 8, 9] having domain $N' = \{\varepsilon : 0 < \varepsilon < \infty\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$\varepsilon^{\lambda} \ln^{r-1} \varepsilon, \quad \ln^r \varepsilon \qquad (\lambda < 0, \quad r \in \mathbb{Z}^+) \tag{3}$$

and all functions of ε which converge to zero in the normal sense as ε tends to zero.

If $f(\varepsilon)$ is a real (or complex) valued function defined on N' and if it is possible to find a constant β such that $f(\varepsilon) - \beta$ is in N, then β is called the neutrix limit of $f(\varepsilon)$ as $\varepsilon \to 0$ and we write $\mathrm{N}{-}\lim_{\varepsilon\to 0} f(\varepsilon) = \beta$.

Note that if a function $f(\varepsilon)$ tends to β in the normal sense as ε tends to zero, it converges to β in the neutrix sense.

On using equation (2), the incomplete gamma function $\gamma(\alpha, x)$ was also defined by

$$\gamma(\alpha, x) = \mathrm{N}{-}\lim_{\varepsilon\to 0} \int_{\varepsilon}^{x} u^{\alpha-1} e^{-u}\, du$$

for all $\alpha \in \mathbb{R}$ and $x > 0$, and it was shown that $\lim_{x\to\infty} \gamma(-m, x) = \Gamma(-m)$ for $m \in \mathbb{N}$, see [4, 9].

The r-th derivative of $\gamma(\alpha, x)$ was similarly defined by

$$\gamma^{(r)}(\alpha, x) = \mathrm{N}{-}\lim_{\varepsilon\to 0} \int_{\varepsilon}^{x} u^{\alpha-1} \ln^r u e^{-u}\, du$$

for all α and $r = 0, 1, 2, \ldots$, provided that the neutrix limit exists, see [9].

The incomplete gamma function with negative arguments are difficult to compute, see [5]. In [10] Thompson gave the algorithm for accurately computing the incomplete gamma function $\gamma(\alpha, x)$ in the cases where $\alpha = n + 1/2, n \in \mathbb{Z}$ and $x < 0$.

However, it was pointed out in [3] that equation (1) could be replaced by the equation

$$\gamma(\alpha, x) = \int_0^x |u|^{\alpha-1} e^{-u}\, du \tag{4}$$

and this equation was used to define $\gamma(\alpha, x)$ for all x and $\alpha > 0$, the integral again diverging for $\alpha \leq 0$.

2. The Locally Summable Function $\gamma_*(\alpha, x_-)$

The locally summable function $\gamma_*(\alpha, x_-)$ is defined on the real line for $\alpha > 0$ by

$$\begin{aligned} \gamma_*(\alpha, x_-) &= \begin{cases} \int_0^x |u|^{\alpha-1} e^{-u}\, du, & x \leq 0, \\ 0, & x > 0 \end{cases} \\ &= \int_0^{-x_-} |u|^{\alpha-1} e^{-u}\, du \end{aligned} \tag{5}$$

see [3, 8] and can be defined as a distribution for $\alpha < 0$ and $\alpha \neq -1, -2, \ldots$ by recurrence formula

$$\gamma_*(\alpha+1, x_-) = -\alpha\gamma_*(\alpha, x_-) - x_-^{\alpha} e^{-x}. \tag{6}$$

If $-m < \alpha < -m+1, m \in \mathbb{N}$ it is defined by

$$\gamma_*(\alpha, x_-) = \int_0^{-x_-} |u|^{\alpha-1} \Big[e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!}\Big] du - \sum_{i=0}^{m-1} \frac{x_-^{\alpha+i}}{(\alpha+i)i!}. \tag{7}$$

It was noted in [8] that the function $\gamma_*(\alpha, x_-)$ can be defined by

$$\gamma_*(\alpha, x_-) = \operatorname*{N-lim}_{\varepsilon\to 0} \int_{-\varepsilon}^{-x_-} |u|^{\alpha-1} e^{-u}\, du$$

and this suggusted that the incomplete gamma type function $\gamma_*(-m, x_-)$ be defined by

$$\gamma_*(-m, x_-) = \operatorname*{N-lim}_{\varepsilon\to 0} \int_{-\varepsilon}^{-x_-} |u|^{-m-1} e^{-u}\, du \tag{8}$$

for $x < 0$ and $m \in \mathbb{N}$. Using equation (7) and taking the neutrix limit, it was shown that

$$\begin{aligned}\gamma_*(-m, x_-) &= \operatorname*{N-lim}_{\varepsilon\to 0} \int_{-\varepsilon}^{-x_-} |u|^{-m-1} e^{-u}\, du \\ &= \int_0^{-x_-} |u|^{-m-1}\Big[e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!}\Big]\, du - \sum_{i=0}^{m-1} \frac{x_-^{i-m}}{(m-i)i!} - \frac{1}{m!}\ln x_-\end{aligned} \tag{9}$$

and also written in the form

$$\begin{aligned}\gamma_*(-m, x_-) \quad = \quad & \int_0^{-1} |u|^{-m-1}\Big[e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!}\Big]\, du \\ & + \int_{-1}^{-x_-} |u|^{-m-1} e^{-u}\, du + \sum_{i=0}^{m-1} \frac{1}{(m-i)i!}.\end{aligned} \tag{10}$$

If $m = 0$, then

$$\begin{aligned}\gamma_*(0, x_-) \quad &= \quad \operatorname*{N-lim}_{\varepsilon\to 0} \int_{-\varepsilon}^{-x_-} |u|^{-1} e^{-u}\, du \\ &= \quad \int_0^{-x_-} |u|^{-1}(e^{-u} - 1)\, du - \ln x_-.\end{aligned} \tag{11}$$

Taking the derivative of $\gamma_*(\alpha, x_-)$, we have

$$\gamma_*^{(r)}(\alpha, x_-) = \operatorname*{N-lim}_{\varepsilon\to 0} \int_{-\varepsilon}^{-x_-} |u|^{\alpha-1} \ln^r |u| e^{-u}\, du \tag{12}$$

for all $\alpha < 0, r = 0, 1, 2, \ldots$ and $x < 0$.

The distribution x_-^{-m} is defined by

$$x_-^{-m} = -\frac{1}{(m-1)!}(\ln x_-)^{(m)}.$$

The definition of x_-^{-m} here is not the same as Gelfand and Shilov's definition of x_-^{-m} which we will denote by $F(x_-, -m)$ and it is shown that

$$x_-^{-m} = F(x_-, -m) + \frac{\phi(m-1)}{(m-1)!}\delta^{(m-1)}(x) \tag{13}$$

for $m = 1, 2, \ldots$, see [2], where

$$\phi(m) = \begin{cases} 0, & m = 0, \\ \sum_{i=1}^{m} i^{-1}, & m > 0.\end{cases}$$

The following two equations are easily satisfied;

$$\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}|x|^{\alpha}\varphi(x)\,dx=\langle x_-^{\alpha},\varphi(x)\rangle \tag{14}$$

if $-m-1<\alpha<-m$ for $m=1,2,\ldots$ and

$$\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}x^{-m}\varphi(x)\,dx=(-1)^m\langle F(x_-,-m),\varphi(x)\rangle \tag{15}$$

for arbitrary $\varphi\in\mathcal{D}$ and $m=1,2,\ldots$.

In fact, we have

$$\begin{aligned}\int_{-\infty}^{-\varepsilon}|x|^{\alpha}\varphi(x)\,dx &= \int_{-\infty}^{-\varepsilon}|x|^{\alpha}\Big[\varphi(x)-\sum_{i=0}^{m-1}\frac{\varphi^{(i)}(0)}{i!}x^i\Big]dx+\sum_{i=0}^{m-1}\frac{\varphi^{(i)}(0)}{i!}\int_{-\infty}^{-\varepsilon}(-x)^{\alpha+i}\\ &= \int_{-\infty}^{-\varepsilon}|x|^{\alpha}\Big[\varphi(x)-\sum_{i=0}^{m-1}\frac{\varphi^{(i)}(0)}{i!}x^i\Big]dx-\sum_{i=0}^{m-1}\frac{\varphi^{(i)}(0)\varepsilon^{\alpha+i+1}}{(\alpha+i+1)i!}\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}|x|^{\alpha}\varphi(x)\,dx &= \int_{-\infty}^{0}|x|^{\alpha}\Big[\varphi(x)-\sum_{i=0}^{m-1}\frac{\varphi^{(i)}(0)}{i!}x^i\Big]dx\\ &= \langle |x|^{\alpha},\varphi(x)\rangle\end{aligned}$$

proving equation(14).

Similarly

$$\begin{aligned}\int_{-\infty}^{-\varepsilon}x^{-m}\varphi(x)\,dx &= \int_{-\infty}^{-\varepsilon}x^{-m}\Big[\varphi(x)-\sum_{i=0}^{m-2}\frac{\varphi^{(i)}(0)}{i!}x^i-H(x+1)\frac{\varphi^{(m-1)}(0)x^{m-1}}{m-1)!}\Big]dx\\ &\quad+\sum_{i=0}^{m-2}\frac{\varphi^{(i)}(0)}{i!}\int_{-\infty}^{-\varepsilon}x^{-m+i}\,dx+\frac{\varphi^{(m-1)}(0)}{(m-1)!}\int_{-\infty}^{-\varepsilon}x^{-1}\,dx\end{aligned}$$

and it follows that

$$\begin{aligned}&\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}x^{-m}\varphi(x)\,dx=\\ &=\int_{-\infty}^{0}x^{-m}\Big[\varphi(x)-\sum_{i=0}^{m-2}\frac{\varphi^{(i)}(0)}{i!}x^i-H(x+1)\frac{\varphi^{(m-1)}(0)x^{m-1}}{m-1)!}\Big]dx\\ &=(-1)^m\langle F(x_-,-m),\varphi(x)\rangle\end{aligned}$$

proving equation (15).

The following theorem was given in [3].

Theorem 2.1.

$$\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}|u|^{\alpha}\,du\,dx=-\frac{\langle x_-^{\alpha+1},\varphi(x)\rangle}{\alpha+1} \tag{16}$$

if $-m-1<\alpha<-m,\ m=1,2,\ldots$ *and*

$$\mathrm{N-}\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}u^{-1}\,du\,dx=\langle \ln x_-,\varphi(x)\rangle, \tag{17}$$

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}u^{-m}\,du\,dx &= \frac{\langle F(x_-,-m+1),\varphi(x)\rangle}{m-1} \\
&\quad+\frac{(-1)^m\langle\delta^{(m-2)}(x),\varphi(x)\rangle}{(m-1)(m-1)!} \\
&= \frac{\langle x_-^{-m+1},\varphi(x)\rangle}{m-1} \\
&\quad+\frac{(-1)^m\phi(m-1)\langle\delta^{(m-2)}(x),\varphi(x)\rangle}{(m-1)(m-1)!} \qquad (18)
\end{aligned}$$

for $m = 2, 3, \dots$ and arbitrary $\varphi \in \mathcal{D}$.

Equations (7) and (16) suggest that the distribution $\gamma_*(\alpha, x_-)$ can be defined by

$$\langle\gamma_*(\alpha,x_-),\varphi(x)\rangle = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}|u|^{\alpha-1}e^{-u}\,du\,dx \qquad (19)$$

if $-m-1 < \alpha < -m$ for $m = 1, 2, \dots$ and $\varphi \in \mathcal{D}$.

As consequence of equation (19), we define $\gamma_*(-m, x_-)$ as follows.

Definition 2.2. *The distribution $\gamma_*(-m, x_-)$ is defined by*

$$\langle\gamma_*(-m,x_-),\varphi(x)\rangle = \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}|u|^{-m-1}e^{-u}\,du\,dx$$

for $m = 1, 2, \dots$ and $\varphi \in \mathcal{D}$.

Theorem 2.3.

$$\begin{aligned}
\langle\gamma_*^{(r)}(\alpha,x_-),\varphi(x)\rangle &= (-1)^r\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi(x)\int_{-\varepsilon}^{-x_-}|u|^{\alpha-1}e^{-u}\,du\,dx \\
&= (-1)^r\langle\gamma_*(\alpha,x_-),\varphi^{(r)}(x)\rangle
\end{aligned}$$

if $-m-1 < \alpha < -m$ for $m = 1, 2, \dots$ and $\varphi \in \mathcal{D}$.

Proof.

$$\begin{aligned}
&(-1)^r\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi^{(r)}(x)\int_{-\varepsilon}^{-x_-}|u|^{\alpha-1}e^{-u}\,du\,dx = \\
&\quad= (-1)^r\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varphi^{(r)}(x)\int_{-\varepsilon}^{-x_-}|u|^{\alpha-1}\Big[e^{-u}-\sum_{i=0}^{m-1}\frac{(-u)^i}{i!}\Big]\,du\,dx \\
&\qquad+(-1)^r\,\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\sum_{i=0}^{m-1}\int_{-\infty}^{-\varepsilon}\frac{\big[\varepsilon^{\alpha+i}-x_-^{\alpha+i}\big]}{(\alpha+i)i!}\varphi^{(r)}(x)\,du\,dx
\end{aligned}$$

On using Taylor's theorem we have

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{\varepsilon\to 0}\int_{-\infty}^{-\varepsilon}\varepsilon^{\alpha+i}\varphi^{(r)}(x)\,dx = \\
&\quad= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\varepsilon^{\alpha+i}[\psi(-\varepsilon)-\psi(-\infty)] \\
&\quad= \mathrm{N}\!-\!\lim_{\varepsilon\to 0}\varepsilon^{\alpha+i}\sum_{j=0}^{m-2}\frac{(-\varepsilon)^j\psi^{(j)}(0)}{j!}+(-1)^{m-1}\lim_{\varepsilon\to 0}\frac{\varepsilon^{m+\alpha}\varphi^{(m-1)}(-\xi\varepsilon)}{(m-1)!} \\
&\quad= 0
\end{aligned}$$

where $\psi(x)$ is the primitive of $\varphi^{(r)}(x)$. Thus

$$\begin{aligned}
&(-1)^r \operatorname{N-lim}_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \varphi^{(r)}(x) \int_{-\varepsilon}^{-x_-} |u|^{\alpha-1} e^{-u}\, du\, dx = \\
&\quad = (-1)^r \operatorname{N-lim}_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \varphi^{(r)}(x) \int_{-\varepsilon}^{-x_-} |u|^{\alpha-1} \Big[e^{-u} - \sum_{i=0}^{m-1} \frac{(-u)^i}{i!}\Big]\, du\, dx \\
&\qquad\qquad - (-1)^r \sum_{i=0}^{m-1} \frac{1}{(\alpha+i)i!} \langle x_-^{\alpha+i}, \varphi^{(r)}(x)\rangle \\
&\quad = (-1)^r \langle \gamma_*(\alpha, x_-), \varphi^{(r)}(x)\rangle.
\end{aligned}$$

□

Theorem 2.3 suggests the following definition.

Definition 2.4. *The distribution $\gamma_*^{(r)}(-m, x_-)$ is defined by*

$$\begin{aligned}
\langle \gamma_*^{(r)}(-m, x_-), \varphi(x)\rangle &= (-1)^r \operatorname{N-lim}_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} \varphi(x) \int_{-\varepsilon}^{-x_-} |u|^{-m-1} e^{-u}\, du\, dx \\
&= (-1)^r \langle \gamma_*(-m, x_-), \varphi^{(r)}(x)\rangle
\end{aligned}$$

for arbitrary $\varphi \in \mathcal{D}$ and $r, m = 1, 2, \dots$.

Theorem 2.5. *The following equations*

$$\begin{aligned}
\langle \gamma_*^{(r)}(0, x_-), \varphi(x)\rangle &= (-1)^r \int_{-\infty}^{0} \varphi^{(r)}(x) \int_{0}^{-x_-} |u|^{-1}(e^{-u} - 1)\, du \\
&\qquad - (-1)^r \langle \ln x_-, \varphi^{(r)}(x)\rangle \\
&= (-1)^r \langle \gamma_*(0, x_-), \varphi^{(r)}(x)\rangle \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&\langle \gamma_*^{(r)}(-m, x_-), \varphi(x)\rangle = (-1)^r \langle \gamma_*(-m, x_-), \varphi^{(r)}(x)\rangle \\
&\quad = (-1)^r \int_{-\infty}^{0} \varphi^{(r)}(x) \int_{0}^{-x_-} |u|^{-m-1} \Big[e^{-u} - \sum_{i=0}^{m} \frac{(-u)^i}{i!}\Big]\, du\, dx \\
&\qquad - (-1)^r \sum_{i=0}^{m-1} \frac{1}{(m-i)i!} \langle F(x_-, -m+i), \varphi^{(r)}(x)\rangle \\
&\qquad\quad - (-1)^r \sum_{i=0}^{m-1} \frac{\phi(m-i-1)}{(m-i)!i!} \langle \delta^{(m-i-1)}(x), \varphi^{(r)}(x)\rangle \\
&\qquad\qquad - \frac{(-1)^r}{m!} \langle \ln x_-, \varphi^{(r)}(x)\rangle \qquad (21)
\end{aligned}$$

hold for arbitrary $\varphi \in \mathcal{D}$ and $m = 1, 2, \dots$ and $r = 0, 1, 2, \dots$.

Proof. Equation (20) follows from equation (11) and Definition 2. Similarly Equation (21) follows from equations (9) and (13) and Definition 2.4. □

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